Two-point correlations of the Gaussian symplectic ensemble from periodic orbits

2000 J. Phys. A: Math. Gen. 33 L503

(http://iopscience.iop.org/0305-4470/33/50/102)

J. Phys. A: Math. Gen. **33** (2000) L503–L507. Printed in the UK PII: S0305-4470(00)17396-2

LETTER TO THE EDITOR

Two-point correlations of the Gaussian symplectic ensemble from periodic orbits

Stefan Keppeler

Abteilung Theoretische Physik, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany

E-mail: kep@physik.uni-ulm.de

Received 22 September 2000, in final form 13 October 2000

Abstract. We determine the asymptotics of the two-point correlation function for quantum systems with half-integer spin which show chaotic behaviour in the classical limit using a method introduced by Bogomolny and Keating. For time-reversal-invariant systems we obtain the leading terms of the two-point correlation function of the Gaussian symplectic ensemble. Special attention has to be paid to the rôle of Kramers' degeneracy.

Understanding correlations of energy levels of quantum mechanical systems whose classical limit exhibits chaotic motion is one of the major topics in quantum chaos. The bridge between quantum mechanics and classical mechanics is provided by the Gutzwiller trace formula [1], which relates the quantum mechanical density of states $d(E) = \sum_n \delta(E - E_n)$ to a sum over periodic orbits of the corresponding classical system,

$$d(E) \sim \bar{d}(E) + \frac{1}{2\pi\hbar} \sum_{\gamma} \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathcal{A}_{\gamma k}\, T_\gamma\, \exp\left[\frac{\mathrm{i}}{\hbar} k S_\gamma(E)\right] \qquad \hbar \to 0 \tag{1}$$

where $\bar{d}(E)$ denotes the mean spectral density (which, by Weyl's law, is of the order of $\hbar^{-f}$ for systems with f degrees of freedom), and the sum extends over all primitive periodic orbits γ and their repetitions, formally including negative ones. $S_\gamma(E) = \oint_\gamma \vec{p}\,\mathrm{d}\vec{x}$ denotes the classical action, T_γ is the (primitive) period, $T_\gamma = \mathrm{d}S_\gamma(E)/\mathrm{d}E$, and the amplitude $\mathcal{A}_{\gamma k}$ involves topological and stability properties. The conjecture of Bohigas, Giannoni and Schmit (BGS) [2] states that for classically chaotic systems, generically, the statistics of energy levels can be modelled by the average behaviour of ensembles of random matrices. In the case of systems without spin the relevant ensembles are the Gaussian orthogonal and the Gaussian unitary ensemble (GOE/GUE), depending on whether the system does or does not possess an antiunitary symmetry such as time reversal (see e.g. [3]). In the case of time-reversal-invariant systems with half-integer spin one also has to deal with the Gaussian symplectic ensemble (GSE).

The main result in understanding eigenvalue correlations in terms of the underlying classical dynamics is due to Berry [4]. He used the so-called diagonal approximation and the Hannay–Ozorio de Almeida sum rule [5] (see also [6]) to determine the asymptotics of the spectral form factor, which is the Fourier transform of the two-point correlation function $R_2(s)$ (see equation (6) below). This treatment has recently been generalized to the case with half-integer spin [7] using an analogue of the Gutzwiller trace formula for which the amplitudes $\mathcal{A}_{\gamma k}$ in (1) include an additional factor due to spin precession [8, 9].

In the case of the GOE and the GUE Bogomolny and Keating [10] (see also [11, 12]) developed a method for the semiclassical evaluation of $R_2(s)$, which yields an additional term as compared to the diagonal approximation of the form factor. More precisely, their method yields the leading non-oscillatory and the leading oscillatory term of $R_2(s)$ as $s \to \infty$, whereas the diagonal approximation of the form factor corresponds to the leading non-oscillatory term. Recently Haake [3] proposed a method to adapt this result to the case of the GSE, but, surprisingly, although he obtained two terms of the large-s asymptotics of $R_2(s)$, the method failed to reproduce the leading term. The aim of this Letter is to present a slightly different approach to systems with half-integer spin, which correctly yields both the leading non-oscillatory and the leading oscillatory term. Note that [10] also includes a remark on GSE asymptotics, which, however, is not based on semiclassics with spin but on a theorem in random matrix theory and, therefore, is not to be confused with the problem addressed here.

The general method of [10] consists of three main steps. Starting from the observation that trace formulae lead to accurate semiclassical quantization conditions Bogomolny and Keating propose to base the semiclassical analysis of spectral correlations on such an approximate spectrum. In the course of the calculations they, secondly, employ the diagonal approximation as introduced in [4, 5]. Finally, they make use of the assumption that the oscillating part of the integrated spectral density (i.e. the contribution of periodic orbits) behaves like a Gaussian random variable. Here we will only briefly sketch the necessary changes to the method of Bogomolny and Keating in order to take care of the situation with half-integer spin. For the general formalism we refer to [3, 10–12]. We will also rely heavily on results of [7].

In order to obtain a simple but efficient semiclassical quantization condition, we first integrate (1) over the energy E, which yields a trace formula for the spectral staircase function $N(E)$. Taking into account only orbits up to a time T, which below will be chosen to be of the order of the Heisenberg time $T_{\mathrm{H}} = 2\pi\hbar\bar{d}(E)$, one obtains a truncated spectral staircase function $N_{\mathrm{T}}(E)$, and the semiclassical eigenvalues $E_n(T)$ can be determined from the condition [13, 14]

$$N_{\mathrm{T}}(E_n(T)) \stackrel{!}{=} n + \tfrac{1}{2}. \tag{2}$$

The trace formula (1) can easily be integrated if there is a one-to-one correspondence between orbits at different energies (i.e. no bifurcations occur when varying E), and from successive integration by parts we see that to leading order in $\hbar$ it is sufficient to integrate the exponential, i.e.

$$N_{\mathrm{T}}(E) \sim \bar{N}(E) + \sum_{\gamma} \sum_{\substack{k\in\mathbb{Z}\setminus\{0\} \\ |k|T_\gamma \leqslant T}} \frac{1}{2\pi \mathrm{i}k} \mathcal{A}_{\gamma k} \exp\left[\frac{\mathrm{i}}{\hbar} k S_\gamma(E)\right] \qquad \hbar \to 0 \tag{3}$$

where the periodic orbit sum will later be denoted by $N_{\mathrm{T}}^{\mathrm{fluc}}(E)$. At this point it is important to take care of Kramers' degeneracy. If the quantum system, with Hamiltonian $\hat{H}$, has half-integer spin and is invariant under time reversal, i.e. $[\hat{H}, \hat{T}] = 0$ with $\hat{T} = \mathrm{i}\sigma_y\hat{K}$, where $\hat{K}$ is the operator of complex conjugation, then each energy eigenvalue has multiplicity of at least two. One could now attempt to first calculate the correlations for the degenerate spectrum and relate the result to the correlations of the non-degenerate spectrum (cf [3]). This strategy is successful for the form factor [7]. However, since the truncated spectral staircase function $N_{\mathrm{T}}(E)$ fails to reproduce sharp steps of size two, the quantization condition (2) will yield not degenerate eigenvalues but two distinct eigenvalues, which both have an additional error. Therefore, we instead take Kramers' degeneracy into account at this point by imposing the modified quantization condition

$$N_{\mathrm{T}}(E_n(T)) \stackrel{!}{=} 2n + 1 \tag{4}$$

which produces a semiclassical spectrum $\{E_n(T)\}$ with Kramers' degeneracy already removed. Note that this semiclassical spectrum has mean density $\bar{d}(E)/2$, and the corresponding Heisenberg time is $T_{\mathrm{H}} = \pi\hbar\bar{d}(E)$. From here on we closely follow [10]. Using the Poisson summation formula, the density of states $\tilde{d}(E)$ of the semiclassical spectrum can be written as

$$\tilde{d}(E) := \sum_n \delta(E - E_n(T)) = \tfrac{1}{2} d_{\mathrm{T}}(E) \sum_{\nu\in\mathbb{Z}} (-1)^\nu \exp[\mathrm{i}\pi\nu N_{\mathrm{T}}(E)] \tag{5}$$

where $d_{\mathrm{T}}(E) = \mathrm{d}N_{\mathrm{T}}(E)/\mathrm{d}T$, see (3).

Before we can compare spectral correlations with results from random matrix theory (RMT) the spectrum has to be unfolded, i.e. the eigenenergies are rescaled such that their mean separation is unity. Since our treatment also has to cover non-scaling systems such as spectra of Dirac Hamiltonians, where typically a continuous spectrum is present and the eigenvalues are confined to the gap $(-mc^2, mc^2)$, we have to apply a method of unfolding which is slightly different (although asymptotically equivalent) from the one usually used. To this end consider the spectral interval $I = I(E,\hbar) := [E - \hbar\omega, E + \hbar\omega]$, which contains N_I levels. In the semiclassical limit this number can be estimated by $N_I \sim 2\hbar\omega\bar{d}/2$, where $\bar{d} = \bar{d}(E)$, i.e. as $\hbar \to 0$ the length of the interval shrinks to zero but the number of eigenvalues contained in $I(E,\hbar)$ goes to infinity (cf [7]). Defining the unfolded eigenvalues by $x_n(T) := E_n(T)\bar{d}/2$, the density of states $D_{\mathrm{T}}(x) = \sum_n \delta(x - x_n(T))$, $x = E\bar{d}/2$, of the unfolded spectrum reads $D_{\mathrm{T}}(x) = 2\tilde{d}_{\mathrm{T}}(E)/\bar{d}$. Using (5) the semiclassical two-point correlation function, expressed in terms of the original variable E, is given by (cf equation (7) in [10])

$$\begin{aligned} R_2(s, I) := \frac{1}{\bar{d}^2}\Bigg\langle d_{\mathrm{T}}\left(E' + \frac{s}{\bar{d}}\right) d_{\mathrm{T}}\left(E' - \frac{s}{\bar{d}}\right) \\ \times \sum_{\nu,\nu'\in\mathbb{Z}} (-1)^{\nu-\nu'} \exp\left[\mathrm{i}\pi\left(\nu N_{\mathrm{T}}\left(E' + \frac{s}{\bar{d}}\right) - \nu' N_{\mathrm{T}}\left(E' - \frac{s}{\bar{d}}\right)\right)\right]\Bigg\rangle_{E'} - 1 \end{aligned} \tag{6}$$

where the brackets denote an average over $I(E,\hbar)$, i.e. $\langle\ldots\rangle_{E'} = \frac{1}{2\hbar\omega}\int_{E-\hbar\omega}^{E+\hbar\omega} \cdots \,\mathrm{d}E'$. From the BGS conjecture we expect that in the semiclassical limit $R_2(s, I)$ converges weakly to the random matrix result, i.e. to $R_2^{\mathrm{GSE}}(s)$ in the case considered here. Following [10] we only aim at providing a periodic orbit theory for this relation in the combined limit

$$s \to \infty \qquad \bar{d} \to \infty \qquad \text{and} \qquad s/\bar{d} \to 0 \tag{7}$$

which will allow expansions in $s/\bar{d}$. Here $\bar{d} \to \infty$ is a consequence of the semiclassical limit and Weyl's law. The asymptotics of the GSE result reads (see e.g. [15])

$$R_2^{\mathrm{GSE}}(s) \sim \frac{\pi}{2}\frac{\cos(2\pi s)}{2\pi s} - \frac{1 + \frac{\pi}{2}\sin(2\pi s)}{(2\pi s)^2} \qquad s \to \infty. \tag{8}$$

By a stationary phase argument one easily sees that the terms with $\nu \neq \nu'$ are of relative order $\mathrm{O}(1/\bar{d})$ in the desired limit (7), i.e. we have $R_2(s, I) \sim \sum_{\nu\in\mathbb{Z}} r_\nu(s, I)$ with

$$\begin{aligned} r_\nu(s, I) := \frac{1}{\bar{d}^2}\Bigg\langle d_{\mathrm{T}}\left(E' + \frac{s}{\bar{d}}\right) d_{\mathrm{T}}\left(E' - \frac{s}{\bar{d}}\right) \\ \times \exp\left[\mathrm{i}\pi\nu\left(N_{\mathrm{T}}\left(E' + \frac{s}{\bar{d}}\right) - N_{\mathrm{T}}\left(E' - \frac{s}{\bar{d}}\right)\right)\right]\Bigg\rangle_{E'} - \delta_{\nu 0}. \end{aligned} \tag{9}$$

Note the different pre-factor in the exponent as compared to the respective formula in [10], which is due to the modified quantization condition (4).

The evaluation of $r_0(s, I)$ is straightforward and will not be shown here. The result corresponds to the usual diagonal approximation of the form factor (cf [3, 10]) and, therefore, in the present situation reads [7]

$$r_0(s, I) \approx -\frac{1}{(2\pi s)^2} \tag{10}$$

which is the leading non-oscillating contribution of $R_2^{\mathrm{GSE}}(s)$ as $s \to \infty$ (cf (8). Here '$\approx$' indicates that (10) is not just an asymptotic relation but we have also used the diagonal approximation, which is assumed to be valid in the combined limit (7). Note that unlike in the GOE and GUE cases (10) is not the leading contribution to $R_2^{\mathrm{GSE}}(s)$ as $s \to \infty$, which instead is oscillatory and, therefore, has to be provided by the following analysis.

Introducing an auxiliary variable s', the contributions $r_\nu(s, I)$, $\nu \neq 0$, can be written as derivatives,

$$r_\nu(s, I) \sim \frac{1}{(\mathrm{i}\pi\nu)^2} \frac{\partial^2}{\partial s \partial s'} \mathrm{e}^{\mathrm{i}\pi\nu(s+s')} \left\langle \exp\left[\mathrm{i}\pi\nu\left(N_{\mathrm{T}}^{\mathrm{fluc}}\left(E' + \frac{s}{\bar{d}}\right) - N_{\mathrm{T}}^{\mathrm{fluc}}\left(E' - \frac{s'}{\bar{d}}\right)\right)\right]\right\rangle_{E'} \Bigg|_{s'=s} \tag{11}$$

where we have expanded the smooth part $\bar{N}(E)$ of $N_{\mathrm{T}}(E)$ in powers of $s/\bar{d}$. The next step lies at the heart of the method of [10]. Assuming that the last exponent in (11) behaves like a Gaussian random variable $G(E')$ with zero mean we can use the identity $\langle \exp[\mathrm{i}G(E')]\rangle_{E'} = \exp[\langle -G^2(E')/2\rangle_{E'}]$ and subsequently evaluate the exponent in the diagonal approximation. This assumption is favoured by a well established conjecture on global eigenvalue correlations [16]. Rather than calculating the square of the exponent and making use of the diagonal approximation in the single terms, as in [10], we prefer to treat the term as a whole, which will allow for an interesting observation subsequent to our main result (14). To this end we employ an expansion in $s/\bar{d}$, yielding

$$N_{\mathrm{T}}^{\mathrm{fluc}}\left(E' + \frac{s}{\bar{d}}\right) - N_{\mathrm{T}}^{\mathrm{fluc}}\left(E' - \frac{s'}{\bar{d}}\right) \sim \sum_\gamma \sum_{\substack{k\in\mathbb{Z}\setminus\{0\} \\ |k|T_\gamma \leqslant T}} \frac{\mathcal{A}_{\gamma k}}{2\pi \mathrm{i} k} \exp\left[\frac{\mathrm{i}}{\hbar} k S_\gamma(E')\right]$$
$$\times \left(\exp\left[\frac{\mathrm{i}}{\hbar} k T_\gamma\left(E'\right)\frac{s}{\bar{d}}\right] - \exp\left[-\frac{\mathrm{i}}{\hbar} k T_\gamma\left(E'\right)\frac{s'}{\bar{d}}\right]\right). \tag{12}$$

Squaring this expression we make use of the diagonal approximation, i.e. we only keep contributions of orbits with like actions, which basically results in taking the modulus square of the addends and multiplying by a factor of two, i.e. the generic multiplicity of periodic orbits in time-reversal invariant systems [4,5]. Since we are dealing with spin-$\frac{1}{2}$ systems, recall that the amplitudes $\mathcal{A}_{\gamma k}$ also include a weight factor due to spin precession along the orbits [9]. The Hannay–Ozorio de Almeida sum rule [5] can be modified to include these factors [7]. Provided that a certain combination of translational and spin dynamics, which is known as a skew product flow, is mixing, it was shown that in the limit (7) the spin contribution is given by an integral over the group $SU(2)$, which yields unity. Therefore, it will not appear in the subsequent discussion (see [7] for details of this calculation). Using these results we obtain

$$\left\langle\left(N_{\mathrm{T}}^{\mathrm{fluc}}\left(E' + \frac{s}{\bar{d}}\right) - N_{\mathrm{T}}^{\mathrm{fluc}}\left(E' - \frac{s'}{\bar{d}}\right)\right)^2\right\rangle_{E'} \approx \frac{g}{\pi^2}\int_0^T \frac{1-\cos(\frac{s+s'}{\hbar\bar{d}}T')}{T'}\,\mathrm{d}T'$$
$$\sim \frac{g}{\pi^2}\log\left(\frac{s+s'}{\hbar\bar{d}}T\right). \tag{13}$$

Upon substituting (13) in (11) we observe that the dominating terms derive from $\nu = \pm 1$ and are given by

$$r_{+1}(s, I) + r_{-1}(s, I) \approx \frac{\hbar\bar{d}}{T}\frac{\cos(2\pi s)}{s}. \tag{14}$$

Since the cut-off time T has to be chosen to be of the order of the Heisenberg time $T_{\mathrm{H}} = \pi\hbar\bar{d}$, it has been argued [3,10] that by setting $T = CT_{\mathrm{H}}$ and comparing to the RMT result the constant C can be determined. However, if in (13) we also keep the sub-leading term of the asymptotic expansion of the cosine integral, which is given by Euler's constant, the value of C is altered. Even worse, corrections of the same order could also arise from non-leading contributions to the sum rule, which, unfortunately, are unknown. Nevertheless, the s-dependence of (14) is not affected by these considerations.

Summarizing, the method of Bogomolny and Keating [10] has been applied to time-reversal-invariant systems with half-integer spin. As in the previously studied cases without spin it correctly reproduces the leading non-oscillatory term (10) and the s-dependence of the leading oscillatory term (14) of the two-point correlation function as $s \to \infty$ (cf (8)). We also remark that in the case of broken time-reversal invariance one has to use the original quantization condition (2) and thus obtains the same asymptotics as in the case without spin [10]. These results give further semiclassical evidence for the BGS conjecture, but open questions, such as a consistent determination of the correct cut-off time T, remain.

I would like to thank Professor F Haake for helpful suggestions and providing me with parts of his book manuscript [3] prior to publication, and Professor J P Keating for carefully discussing [10] with me and for many helpful remarks. I am also grateful to Dr J Bolte, R Schubert and Professor F Steiner for useful discussions. This work was partly supported by Deutscher Akademischer Austauschdienst (DAAD) under grant no D/99/02553 and by Deutsche Forschungsgemeinschaft (DFG) under contract no STE 241/10-1.

References

[1] Gutzwiller M C 1971 Periodic orbits and classical quantization conditions *J. Math. Phys.* **12** 343–58
[2] Bohigas O, Giannoni M-J and Schmit C 1984 Characterization of chaotic quantum spectra and universality of level fluctuation laws *Phys. Rev. Lett.* **52** 1–4
[3] Haake F 2000 *Quantum Signatures of Chaos* (Berlin: Springer)
[4] Berry M V 1985 Semiclassical theory of spectral rigidity *Proc. R. Soc.* A **400** 229–51
[5] Hannay J H and Ozorio de Almeida A M 1984 Periodic orbits and a correlation function for the semiclassical density of states *J. Phys. A: Math. Gen.* **17** 3429–40
[6] Parry W and Pollicott M 1990 Zeta functions and the periodic orbit structure of hyperbolic dynamics *Astérisque* **187/188** 1–268
[7] Bolte J and Keppeler S 1999 Semiclassical form factor for chaotic systems with spin 1/2 *J. Phys. A: Math. Gen.* **32** 8863–80
[8] Bolte J and Keppeler S 1998 Semiclassical time evolution and trace formula for relativistic spin-1/2 particles *Phys. Rev. Lett.* **81** 1987–91
[9] Bolte J and Keppeler S 1999 A semiclassical approach to the Dirac equation *Ann. Phys., NY* **274** 125–62
[10] Bogomolny E B and Keating J P 1996 Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation *Phys. Rev. Lett.* **77** 1472–5
[11] Bogomolny E 1998 Spectral statistics *Proc. Int. Congress of Mathematicians (Doc. Math. Extra vol 3)* (Berlin: Springer) pp 99–108
[12] Keating J P 1999 Periodic orbits, spectral statistics, and the Riemann zeros *Superymmetry and Trace Formulae: Chaos and Disorder (NATO ASI Series B: Physics vol 370)* ed I V Lerner, J P Keating and D E Khmelnitskii (New York: Kluwer) pp 1–15
[13] Aurich R and Steiner F 1992 Staircase functions, spectral rigidity, and a rule for quantizing chaos *Phys. Rev.* A **45** 583–92
[14] Aurich R, Matthies C, Sieber M and Steiner F 1992 Novel rule for quantizing chaos *Phys. Rev. Lett.* **68** 1629–32
[15] Mehta M L 1991 *Random Matrices* 2nd edn (San Diego: Academic)
[16] Aurich R, Bolte J and Steiner F 1994 Universal signatures of quantum chaos *Phys. Rev. Lett.* **73** 1356–9